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LETTER TO THE EDITOR

On the product of two Jacobi functions of different kinds with different arguments

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Abstract. We obtain an expansion for the product of two Jacobi functions of different kinds with different arguments in terms of the Jacobi function of the first kind.

Formulae have been given (Watson 1921 and Bailey 1936) which express the product of Jacobi functions of the same kind with different arguments as a sum involving Jacobi functions. As far as we know no attempt has been made to express the product of two Jacobi functions of different kinds with different arguments as a sum involving Jacobi functions. In this paper we show how to obtain this sum by using an integral representation for the product of two Jacobi functions of different kinds with different arguments (Bellandi *et al* 1982). As a by-product we also get sums for Legendre and Gegenbauer functions.

In a recent paper (Bellandi *et al* 1982) we have showed that for the product of $P_\mu^{(\alpha\beta)}(x)$ and $Q_\mu^{(\alpha\beta)}(y)$ there holds the following integral representation,

$$2^{-(\alpha+\beta)}\frac{\Gamma(\mu+1)\Gamma(\mu+\alpha+\beta+1)}{\Gamma(\mu+\alpha+1)\Gamma(\mu+\beta+1)}P_\mu^{(\alpha,\beta)}(x)Q_\mu^{(\alpha,\beta)}(y)$$

$$=(-1)^{\mu+1}\frac{1}{2\pi\mathrm{i}}\int_\infty^{0+}\mathrm{d}v\ (\sinh v)^{-\alpha-\beta-1}\coth^{-(\alpha-\beta)}v/2$$

$$Q^{(\alpha+\beta;0)}\left(2\frac{(x+\cosh v)(y+\cosh v)}{(\cosh v+1)(\cosh v-1)}-1\right) \tag{1}$$

where $\int_\infty^{0+}$ means that the path of integration starts at infinity on the real axis, encircles the origin in the positive sense and returns to the starting point.

Expanding the Jacobi functions on the right-hand side of equation (1) in an absolutely convergent series (Szegö 1939), and interchanging the integral sign with the sum sign, we have

$$2^{-(\alpha+\beta)}\frac{\Gamma(\mu+1)\Gamma(\mu+\alpha+\beta+1)}{\Gamma(\mu+\alpha+1)\Gamma(\mu+\beta+1)}P_\mu^{(\alpha,\beta)}(x)Q_\mu^{(\alpha,\beta)}(y)$$

$$=\tfrac{1}{2}\sum_{\nu=0}^{\infty}(-1)^{\mu+\nu+1}\binom{\mu+\nu}{\nu}\frac{\Gamma(\mu+\nu+\alpha+\beta+1)\,\Gamma(\mu+1)}{\Gamma(2\mu+\nu+\alpha+\beta+2)} \tag{2}$$

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$$\times \frac{1}{2\pi i} \int_{\infty}^{0+} dv \, (\sinh v)^{2\mu+2\nu+\alpha+\beta+1} (\coth v/2)^{-(\alpha-\beta)}$$
$$\times [(x+\cosh v)(y+\cosh v) - \sinh^2 v]^{-\mu-\nu-1-\alpha-\beta}$$

By a single variable transformation we can see that the integral in equation (2) is an analytic continuation of the F_1 Picard function (Erdélyi 1953). Therefore we have

$$2^{-(\alpha+\beta)} \frac{\Gamma(\mu+1)\Gamma(\mu+\alpha+\beta+1)}{\Gamma(\mu+\alpha+1)\Gamma(\mu+\beta+1)} P_\mu^{(\alpha,\beta)}(x) Q_\mu^{(\alpha,\beta)}(y)$$
$$= \frac{1}{2} \sum_{\nu=0}^{\infty} \binom{\mu+\nu}{\nu} \frac{\Gamma(\mu+\nu+\alpha+\beta+1)\Gamma(\mu+1)}{\Gamma(2\mu+\nu+\alpha+\beta+2)} (x+y)^{-\mu-\nu-\alpha-\beta-1} \tag{3}$$
$$\times \frac{\Gamma(\mu+\nu+\alpha+1)}{\Gamma(\alpha+1)\Gamma(\mu+\nu+1)}$$
$$\times F_1\left(-\mu-\nu, -\mu-\nu-\beta, \mu+\gamma+\alpha+\beta+1, \alpha+1; -1, -\frac{1+xy}{x+y}\right).$$

By means of the relation (Erdély 1953)

$$F_1(a, b, b', b+b'; t, z) = (1-z)^{-a} {}_2F_1\left(a, b; b+b'; \frac{z-z}{1-z}\right) \tag{4}$$

we can see that the Picard function in equation (3) is a particular Jacobi function, and therefore we get

$$\frac{\Gamma(\mu+1)\Gamma(\mu+\alpha+\beta+1)}{\Gamma(\mu+\alpha+1)\Gamma(\mu+\beta+1)} P_\mu^{(\alpha,\beta)}(x) Q_\mu^{(\alpha,\beta)}(y)$$
$$= \sum_{\nu=0}^{\infty} \frac{1}{\nu!} 2^{\mu+\nu+\alpha+\beta-1} \frac{\Gamma(\mu+\nu+1)\Gamma(\mu+\nu+\alpha+\beta+1)}{\Gamma(2\mu+\nu+\alpha+\beta+2)} \tag{5}$$
$$\times (x+y)^{-\mu-\nu-\alpha-\beta-1} P_{\mu+\nu}^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right).$$

Since associated Legendre functions are related to Jacobi functions (Erdélyi 1953) by means of

$$P_\mu^{(m,-m)}(x) = \frac{\Gamma(\mu+m+1)}{\Gamma(\mu+1)} \left(\frac{x+1}{x-1}\right)^{m/2} P_\mu^{-m}(x)$$
$$Q_\mu^{(m,-m)}(y) = (-1)^m \frac{\Gamma(\mu-m+1)}{\Gamma(\mu+1)} \left(\frac{y+1}{y+1}\right)^{m/2} Q_\mu^m(y), \tag{6}$$

the corresponding result for associated Legendre functions is

$$P_\mu^{-m}(x)\, Q_\mu^m(y)$$
$$= \sum_{\nu=0}^{\infty} \frac{1}{\nu!} 2^{\mu+\nu-1} \frac{\Gamma(\mu+\nu+1)\Gamma(\mu+\nu+m+1)}{\Gamma(2\mu+\nu+2)} (x+y)^{-\mu-\nu-1} P_{\mu+\nu}^{-m}\left(\frac{1+xy}{x+y}\right). \tag{7}$$

When $m=0$ in equation (7) we have the particular result for Legendre functions.

Gegenbauer functions are particular cases of Jacobi functions (Erdélyi 1953) when $\alpha=\beta=\lambda-\frac{1}{2}$ with $\lambda>\frac{1}{2}$, and the corresponding result is

$$\frac{\Gamma(\frac{1}{2})\Gamma(\mu+1)}{\Gamma(\lambda)\Gamma(\mu+2\lambda)}C_\mu^\lambda(x)D_\mu^\lambda(y)$$

$$=\sum_{\nu=0}^{\infty}\frac{1}{\nu!}2^{\mu+\nu+2\lambda-2}\frac{\Gamma(\mu+\nu+1)\Gamma(\mu+\nu+\lambda+\frac{1}{2})}{\Gamma(2\mu+\nu+2\lambda+1)} \qquad (8)$$

$$\times(x+y)^{-\mu-\nu-2\lambda}C_{\mu+\nu}^\lambda\left(\frac{1+xy}{x+y}\right).$$

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